Homework 2 Solution.

1. Please prove that the addition and scalar multiplication operations of quotient space are well defined. i.e. suppose $\vec{v} + W = \vec{v}' + W$, then for any $\vec{v}'' + W \in V/W$, the following equations hold.

   (a) $(\vec{v} + W) + (\vec{v}'' + W) = (\vec{v}' + W) + (\vec{v}'' + W)$

   (b) $a \cdot (\vec{v} + W) = a \cdot (\vec{v}' + W)$ for any $a \in F$

Proof: $\vec{v} + W = \vec{v}' + W \iff \vec{v} - \vec{v}' \in W$

(a) Since $\vec{v} - \vec{v}' \in W$,

we have $(\vec{v} + \vec{v}'') - (\vec{v}' + \vec{v}'') \in W$

Thus $(\vec{v} + \vec{v}'') + W = (\vec{v}' + \vec{v}'') + W$

i.e. $(\vec{v} + W) + (\vec{v}'' + W) = (\vec{v}' + W) + (\vec{v}'' + W)$

(b) Since $\vec{v} - \vec{v}' \in W$

we have $a\vec{v} - a\vec{v}' = a(\vec{v} - \vec{v}') \in W$

Thus $a\vec{v} + W = a\vec{v}' + W$

i.e $a(\vec{v} + W) = a(\vec{v}' + W)$

2. With the addition and scalar multiplication of quotient space defined in lecture note, show that V/W is a vector space over F.

Proof: $\forall\ v_1+W,\ v_2+W,\ v_3+W \in V/W,\ \forall a, b \in F$

① Closed under addition

$$(v_1+W)+(v_2+W) = (v_1+v_2)+W \in V/W$$

② closed under scalar multiplication

$$a(v_1+W) = av_1 + W \in V/W$$

(VS 1) $(v_1+W)+(v_2+W) = (v_1+v_2)+W$
$= (v_2+v_1)+W$
$= (v_2+W)+(v_1+W)$

(VS2) $[(v_1+W)+(v_2+W)]+(v_3+W) = [(v_1+v_2)+W]+(v_3+W)$
$= (v_1+v_2+v_3)+W$
$= (v_1+W)+[(v_2+v_3)+W]$
$= (v_1+W)+[(v_2+W)+(v_3+W)]$

(VS3) $\exists\ 0_V+W \in V/W$ s.t. $(v_1+W)+(0_V+W) = (v_1+0_V)+W$
$= v_1+W$

(VS4) $\exists\ (-v_1)+W \in V/W$ s.t. $[(-v_1)+W]+(v_1+W) = (-v_1+v_1)+W$
$= 0_V+W$

(VS5) $1\cdot(v_1+W) = (1\cdot v_1)+W = v_1+W$

(VS6) $(ab)(v_1+W) = (abv_1)+W = a((bv_1)+W) = a\cdot[b\cdot(v_1+W)]$

(VS7) $a[(v_1+W)+(v_2+W)] = a\cdot[(v_1+v_2)+W] = a(v_1+v_2)+W$
$= (av_1+av_2)+W = (av_1+W)+(av_2+W)$
$= a(v_1+W)+a(v_2+W)$

(VS8) $(a+b)(v_1+W) = (a+b)v_1+W = (av_1+bv_1)+W$
$= (av_1+W)+(bv_1+W) = a(v_1+W)+b(v_1+W)$

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3. Sec. 1.7: Q7

Prove the following generalization of the replacement theorem. Let β be a basis for a vector space V, and let S be a linearly independent subset of V. There exists a subset S_1 of β such that $S \cup S_1$ is a basis for V.

Proof: Let $\mathcal{F}$ denotes the family of linearly independent subsets of $\beta \cup S$ that contain S.

Claim: For any chain $\mathcal{C}$ in $\mathcal{F}$, there exists a member U of $\mathcal{F}$ such that U contains all members of $\mathcal{C}$.

Let U be the union of all members of $\mathcal{C}$.
Then U contains all members of $\mathcal{C}$.
$\forall\ u_1 \cdots u_n \in U$, $\exists\ A_i \in \mathcal{C}$ st $u_i \in A_i$, $i=1,\cdots,n$
Since $\mathcal{C}$ is a chain, one of these sets, say A_k, contains all the others. So $u_i \in A_k$ $i=1,\cdots,n$
Since $\{u_1,\cdots,u_n\}$ is a subset of lin. ind. A_k, we have $\{u_1 \cdots u_n\}$ is lin. ind.
Since $\{u_1 \cdots u_n\}$ is chosen arbitrarily in U, we have U is lin. ind.
Besides $S \subset U \subset \beta \cup S$.
Thus $U \in \mathcal{F}$.

The maximal principle implies that $\mathcal{F}$ contains a maximal element α, easily seen to be a maximal linearly independent subset of $\beta \cup S$ that contains S.

Let $S_1 = \alpha \setminus S \subset (\beta \cup S) \setminus S \subset \beta$

Then $\alpha = S_1 \cup S$ is a basis for V. (By Thm 1.12)

4. (Extension to Sec. 2.1: Q18) Please find **ALL** linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $N(T) = R(T)$.

A linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ has the general form

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

① By T is linear, we have $T\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Thus $e = f = 0$. $\quad T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

② Since $R(T) = N(T)$. $\quad 2 = \dim(\mathbb{R}^2) = \dim(R(T)) + \dim(N(T))$

Thus $\dim(R(T)) = \dim(N(T)) = 1$

Let $R(T) = N(T) = \left\{ \begin{pmatrix} \lambda t \\ \beta t \end{pmatrix} \in \mathbb{R}^2 : t \in \mathbb{R},\ \lambda^2 + \beta^2 \neq 0 \right\}$

③ Case: $\lambda \neq 0$ and $\beta \neq 0$

$\forall \begin{pmatrix} x \\ y \end{pmatrix} \in N(T)$. $\quad T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

i.e. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \lambda t \\ \beta t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a\lambda + b\beta = 0 \\ c\lambda + d\beta = 0 \end{cases}$

Therefore $a : b = (-\beta) : \lambda = c : d$

Let $\begin{cases} a = -\beta r,\ b = \lambda r. \\ c = -\beta s,\ d = \lambda s \end{cases}$ So $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} -\beta r & \lambda r \\ -\beta s & \lambda s \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$\forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. $\quad T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) \in R(T)$

i.e. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda t \\ \beta t \end{pmatrix} \Rightarrow (ax+by) : (cx+dy) = \lambda : \beta$

$\Rightarrow \beta(ax+by) = \lambda(cx+dy)$

$\beta(-\beta r \cdot x + \lambda r y) = \lambda(-\beta s x + \lambda s y)$

$\beta r(-\beta x + \lambda y) = \lambda s(-\beta x + \lambda y) \quad \forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$

Thus $\beta r = \lambda s$. $\quad r : s = \lambda : \beta$

Let $\begin{cases} r = q \cdot \lambda \\ s = q \cdot \beta \end{cases}$ $q \in \mathbb{R} \setminus \{0\}$

Then $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} -\beta \cdot q\lambda & \lambda q \lambda \\ -\beta \cdot q \cdot \beta & \lambda q \beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$= q \cdot \begin{pmatrix} -\beta\lambda & \lambda^2 \\ -\beta^2 & \lambda\beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ where $\lambda \neq 0$ $\beta \neq 0$ $q \neq 0$

④ Case: $\lambda = 0$, and $\beta \neq 0$

Then $R(T) = N(T) = \left\{ \begin{pmatrix} 0 \\ t \end{pmatrix} \in \mathbb{R}^2 : t \in \mathbb{R} \right\}$

$\forall \begin{pmatrix} x \\ y \end{pmatrix} \in N(T)$ $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

i.e. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow b = d = 0$

So $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$\forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) \in R(T)$

i.e. $\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ t \end{pmatrix}$ for some $t \in \mathbb{R}$.

Thus $a = 0$. So $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$ $c \neq 0$

⑤ Case: $\lambda \neq 0$. and $\beta = 0$

Similar to ④. we have $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ $b \neq 0$.

By Combining ③ ④ ⑤. we have

$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = q \cdot \begin{pmatrix} -\beta\lambda & \lambda^2 \\ -\beta^2 & \lambda\beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ where $\lambda^2 + \beta^2 \neq 0$, $q \neq 0$

5. Sec. 2.1: Q25(c)

Let $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{R}^3$.

(a) If $\mathsf{T}(a,b,c) = (a,b,0)$, show that T is the projection on the xy-plane along the z-axis.

(b) Find a formula for $\mathsf{T}(a,b,c)$, where T represents the projection on the z-axis along the xy-plane.

(c) If $\mathsf{T}(a,b,c) = (a-c,b,0)$, show that T is the projection on the xy-plane along the line $L = \{(a,0,a)\colon a \in R\}$.

(c) For any $(x,y,z) \in \mathbb{R}^3$

$\{(1,0,0),(0,1,0)\}$ is a basis for x-y plane

$\{(1,0,1)\}$ is a basis for L

$$(x,y,z) = \alpha(1,0,0) + \beta(0,1,0) + \gamma(1,0,1)$$

$$\Rightarrow \begin{cases} \alpha+\gamma = x \\ \beta = y \\ \gamma = z \end{cases} \Rightarrow \begin{cases} \alpha = x-z \\ \beta = y \\ \gamma = z \end{cases}$$

i.e. $$(x,y,z) = \underbrace{(x-z)\cdot(1,0,0) + y(0,1,0)}_{\in \text{ x-y plane}} + \underbrace{z(1,0,1)}_{\in L}$$

So $$T((x,y,z)) = (x-z, y, 0)$$

6. Sec. 2.1: Q26

Definition. *Let* V *be a vector space and* W_1 *and* W_2 *be subspaces of* V *such that* $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$*. (Recall the definition of direct sum given in the exercises of Section 1.3.) A function* $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ *is called the* ***projection on*** W_1 ***along*** W_2 *if, for* $x = x_1 + x_2$ *with* $x_1 \in \mathsf{W}_1$ *and* $x_2 \in \mathsf{W}_2$*, we have* $\mathsf{T}(x) = x_1$.

Using the notation in the definition above, assume that $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ is the projection on W_1 along W_2.

(a) Prove that T is linear and $\mathsf{W}_1 = \{x \in \mathsf{V}\colon \mathsf{T}(x) = x\}$.
(b) Prove that $\mathsf{W}_1 = \mathsf{R}(\mathsf{T})$ and $\mathsf{W}_2 = \mathsf{N}(\mathsf{T})$.
(c) Describe T if $\mathsf{W}_1 = \mathsf{V}$.
(d) Describe T if W_1 is the zero subspace.

(a) $\forall a \in F$. $\forall v, v' \in V$. Let $v = \underset{\in W_1}{w_1} + \underset{\in W_2}{w_2}$ $\quad v' = \underset{\in W_1}{w_1'} + \underset{\in W_2}{w_2'}$

$$
\begin{aligned}
T(av+v') &= T(a(w_1+w_2) + (w_1'+w_2')) \\
&= T(\underbrace{(aw_1+w_1')}_{\in W_1} + \underbrace{(aw_2+w_2')}_{\in W_2}) \\
&= aw_1 + w_1' \\
&= a\cdot T(v) + T(v')
\end{aligned}
$$

So T is linear.

① $\forall w \in W_1$, $w = \underset{\in W_1}{w} + \underset{\in W_2}{0}$ So $T(w) = w$. $w \in \{x \in V : T(x) = x\}$

② $\forall w \in \{x \in V : T(x) = x\}$, $w = T(w) \in W_1$

By ① and ②, we have $W_1 = \{x \in V : T(x) = x\}$

(b) $\forall x \in W_1$, $x = \underset{\in W_1}{x} + \underset{\in W_2}{0}$. $x = T(x) \in R(T)$ So $W_1 \subset R(T)$.
By definition. $R(T) \subset W_1$. Thus $W_1 = R(T)$

$\forall y \in W_2$. $y = \underset{\in W_1}{0} + \underset{\in W_2}{y}$. $T(y) = 0$ So $y \in N(T)$, $W_2 \subset N(T)$

$\forall y \in N(T)$ $T(y) = 0$ Let $y = \overset{\in W_1}{y_1} + \overset{\in W_2}{y_2}$ where $y_1 = 0$
Then $y = 0 + y_2 = y_2 \in W_2$. Thus $N(T) \subset W_2$

(c) $T(v) = v$ $\forall v \in V$

(d) $T(v) = \vec{0}_V$ $\forall v \in V$.